Quantum localization for a kicked rotor with accelerator mode islands

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Dynamical localization of classical superdiffusion for the quantum kicked rotor is studied in the semiclassical limit. Both classical and quantum dynamics of the system become more complicated under the conditions of mixed phase space with accelerator mode islands. Recently, long time quantum flights due to the accelerator mode islands have been found. By exploration of their dynamics, it is shown here that the classical–quantum duality of the flights leads to their localization. The classical mechanism of superdiffusion is due to accelerator mode dynamics, while quantum tunneling suppresses the superdiffusion and leads to localization of the wave function. Coupling of the regular type dynamics inside the accelerator mode island structures to dynamics in the chaotic sea proves increasing the localization length. A numerical procedure and an analytical method are developed to obtain an estimate of the localization length which, as it is shown, has exponentially large scaling with the dimensionless Planck's constant $\tilde{h} \ll 1$ in the semiclassical limit. Conditions for the validity of the developed method are specified.

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I. INTRODUCTION

Systems with mixed phase space, where the motion in some regions of phase space is chaotic and in other regions is regular, model many physical problems. A paradigm of such systems is the kicked rotor (standard map). For most values of parameters it was found to exhibit classical diffusion in phase space [1–5]. This diffusion is suppressed by quantum interference, leading to localization in phase space [6], which is similar in nature to Anderson localization in condensed matter physics [7]. For some values of parameters stability islands, namely accelerator mode islands, emerge bifurcatively from marginally stable points inside a chaotic sea [8,9]. These islands correspond to complicated solutions with trajectories that are neither rational nor ballistically monotonic (their geometrical characterization is different from resonant islands in the near integrable limit) [10]. Islands such as there were first considered in Ref. [11] (see also [9]), and for the standard map these were first studied in [8]. The main role of the islands' boundaries for classical transport is their sticky nature, i.e., the long stay of trajectories inside boundary layers. For the general case these accelerator mode islands can coexist with regular or resonant islands. Resonant islands were studied in detail in [12,13]. Unlike the resonant islands, stickiness to boundary layers of the accelerator mode islands leads to superdiffusion. Therefore, for some values of the parameters, influence of the boundary structures dominates the classical transport, which is stronger than diffusion for these values [14–16]. For long time quantum dynamics, the regular parts of the accelerator mode islands dominate the quantum transport. In the present work we will explore how these accelerator mode islands affect the quantum dynamics. We will conclude that localization in phase space also takes place for these values of the parameters, but in a different way. Namely, for quantum localization to take place, quantum tunneling from the inner regular parts of the accelerator mode islands to the chaotic sea is crucial [17]. This tunneling takes place through the boundary structure, which behaves as a potential barrier for long time dynamics. In this case a rate of exponential decay of the initial population or a survival probability due to quantum tunneling determines a rate of localization of the initial wave packet. We argue here that the localization length relates to the coefficient of quantum tunneling. The tunneling mechanism provides an increase of the localization length, which as it is shown here, has exponentially large scaling with the dimensionless Planck's constant in the semiclassical limit.

The kicked rotor can be defined in some units by the Hamiltonian

$$H = p^2/2 + K\cos\theta \sum_{l=-\infty}^{\infty} \delta(t-l), \tag{1.1}$$

where p is the angular momentum, θ is the conjugate angle, and t is the time. The resulting classical motion is given by the standard map

$$p' = p + K\sin\theta,$$
$$\theta' = \theta + p', \tag{1.2}$$

where θ and θ' are the angles at the two consecutive kicks, while p and p' are the values of the angular momentum just before each of these kicks. The classical motion takes place on the infinite cylinder where p is unbounded, while $-\pi \le \theta \le \pi$. For $K \ge K^* = 0.97\ldots$ chaotic regions are connected and for most values of parameters diffusion in momentum takes place, namely

$$\langle p^2 \rangle = D(K)t, \tag{1.3}$$

where $\langle \ldots \rangle$ denotes the average over initial conditions [1–3]. Different classical behavior can be found as well. The marginally stable points are defined by the conditions K_k

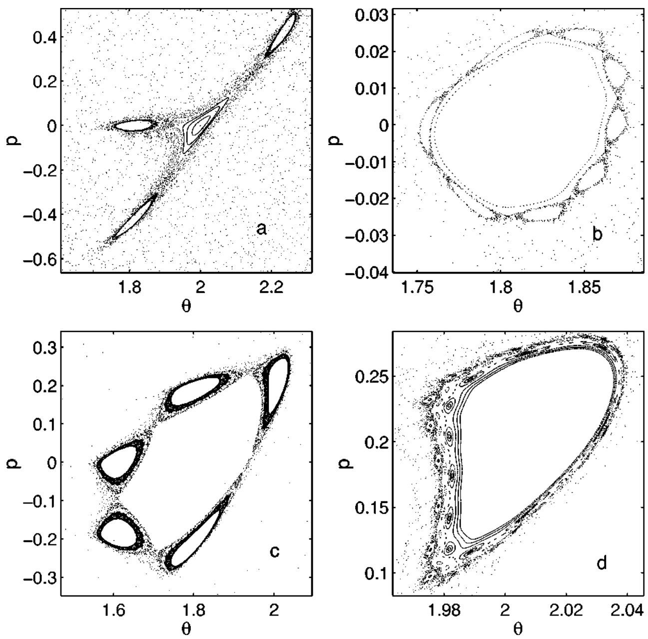

FIG. 1. Phase portrait of the map Eq. (1.2), where (a) and (b) are sequences of island chains of the first and second generations, respectively, with periods 3 and 8 for $K=K^{(1)}$; (c) and (d) are the same as (a) and (b) for $K=K^{(2)}$ with period 5 for the first generation and 11 for the second one.

$=2\pi k$, $(p_0,\theta_0)=(2\pi m,\pm\pi/2)$ with integer (k,m). These points are shifted by a constant value in p at each iteration of the map Eq. (1.2), namely the shift by $\pm 2\pi k$ for $\theta_0=\pm\pi/2$ correspondingly. If initial conditions coincide exactly with these points we encounter the accelerator modes [2] resulting in

$$\langle p^2\rangle \sim t^2. \tag{1.4}$$

In the vicinity ΔK_k of such values K for

$$0<K-K_k<\Delta K_k \tag{1.5}$$

a new set of islands is generated [8,10,14], called tangled islands in [10], as a result of bifurcation. Dynamics inside the islands is known as the accelerator mode [17,14]. Changes of K within the interval Eq. (1.5) strongly influence the topological structures of the accelerator mode islands. Examples of such structures are presented in Fig. 1. Boundaries of these islands are complicated structures which can exhibit self-similar infinite hierarchical island chains for some specific values of the parameter K that are known as "magic" numbers. For example, for $K=K^{(1)}=6.908\,745\ldots$ the number of islands in the various generations (in increasing order) is $(3,8,8,8,\ldots)$ [14], while for $K=K^{(2)}=6.476\,939\ldots$ it is $(5,11,11,11,\ldots)$ [15,16] (see Fig. 1). Trajectories with initial points in the inner regular region are trapped inside and in each step of the map their momentum increases by 2π, resulting in acceleration and a growth of the momentum of the form Eq. (1.4). Because of the island chains these structures also affect trajectories that are started in the chaotic region. Such trajectories are trapped for some time in the island chains and their propagation in phase space is faster than diffusion [14–16]. This results in anomalous diffusion, namely

$$\langle p^2\rangle \sim t^{\mu} \tag{1.6}$$

with $\mu>1$. This anomalous diffusion can be found for a large variety of narrow regions of K (see Fig. 8 in Ref. [14]).

The diffusive classical behavior is suppressed by quantum interference [6,7,18]. To study its effects it is useful to introduce the evolution operator

$$\hat{U}=\exp\{-i\tilde{h}\hat{n}^2/2\}\exp\{-i(K/\tilde{h})\cos\theta\}, \tag{1.7}$$

where $\tilde{h}$ is the dimensionless Planck's constant, and the angular momentum in these units is $\hat{p}=\tilde{h}\hat{n}$, with $\hat{n}=-i\partial/\partial\theta$. The eigenvalues of $\hat{U}$ are $e^{-i\omega}$, where the real number ω is the quasienergy and the corresponding eigenstates are the quasienergy states.

Suppression of diffusion means that the quasienergy states are localized in angular momentum. For most values of $K>K^*$ these were found to be exponentially localized and for large K and small $\tilde{h}$ the localization length in n is [19,20]

$$\xi=D(K)/2\tilde{h}^2. \tag{1.8}$$

Deviations were found near accelerator modes and the "magic" numbers [21–24]. There is an infinite set of so–called "magic" numbers for which the diffusion is anomalous [14]. The existence of localization and its nature were questioned for these values of K [17,21,22,25–27]. This question is relevant for experiments on localization in linear momentum of driven laser cooled atoms [28].

The purpose of the present work is to study localization for such "magic " values of K. For the complete quantum dynamics the $\tilde{h}$ scaling according to Eq. (1.8) is relevant [21,26]. Nevertheless, influence of the accelerator mode islands on localization has been observed in a change of the shape of the stationary distribution probability over momentum states for these values of $\tilde{h}$ larger than the area of the islands. Deviation from the scaling of Eq. (1.8) was also observed [22] when the quantum parameter $\tilde{h}$ decreases and becomes compareable with the area of the accelerator mode island structure. We show here that the presence of the accelerator mode islands is crucial for semiclassical analysis of quantum localization of anomalous diffusion. The main quantum effect that has to be considered is tunneling. As a result of tunneling, the occupation of the accelerator mode decays exponentially [17,25]. For the boundary islands chain of the accelerator mode, gaps between islands behave as effective barriers if these become smaller than the de Broglie wavelength, resulting in the crossover between quantum and classical trapping [29–34] (see detailed discussion in Ref. [31] and references therein). In Sec. II a variant of the quantum map generated by Eq. (1.7) will be constructed for numerical exploration of the quantum effects on transport for values of K where phase space structures like those of Fig. 1 are expected to be important. These structures should be of a substantial size on the scale of the Planck's constant $\tilde{h}$. In Sec. III the tunneling rate will be estimated analytically and the analytical predictions of dynamics for the survival probability will be compared with the numerical results. The coefficients of tunneling will be obtained numerically for the two different values of K related to the structures shown in Fig. 1. In Sec. IV the results on the transport will be used to develop an estimate for the localization length to specify what is the difference compared to the formula (1.8) for these special values of K. It will be shown that in the framework of the semiclassical considerations the localization length increases exponentially with $\tilde{h}$. For other values of K localization, that was found in the past, as expected. Conditions when Eq. (1.8) is valid for localization of anomalous diffusion have been well discussed (see Refs. [21,22] and [26]). The summary of the analysis is presented in Sec. V.

II. QUANTUM TRANSPORT FOR THE ACCELERATOR MODE STRUCTURES: NUMERICAL CALCULATIONS

In this section the long time behavior of the survival probability of a quantum particle in an accelerator mode island is studied numerically. In what follows, we call this type of solution an accelerator mode island structure or, for brevity, structure, to stress that for quantum dynamics, both boundary layers and regular parts of the accelerator mode islands are important. Moreover, as it will be seen, the regular part of the islands plays the role of a potential well for the quantum wave packet, while the island boundary layers behave as a potential barrier through which the wave packet tunnels. Numerical study of the problem is based on investigation of the quantum survival probability in some domain $\Delta p \in (-\pi,\pi)$ that includes a structure like those shown in Fig. 1. The main problem in the numerical exploration results from the fact that a part of the wave function that belongs to the islands structures propagates as an accelerator mode along p. A simple way to avoid this type of escape from Δp and to calculate the probability of the particle remaining inside the structure, ignoring the parts that leak out from Δp, is to shift the structure to its original position. A shift by any $-n_0$ in momentum is induced by the shift operator

$$\hat{J}=\exp(-in_0\,\theta), \tag{2.1}$$

which application to an arbitrary wave function $\Psi(\theta)=\Sigma_n f_n e^{in\theta}$ gives

$$\hat{J}\Psi(\theta)=\sum_n f_n\, e^{i(n-n_0)\theta}=\sum_n f_{n+n_0}\, e^{in\theta}. \tag{2.2}$$

Therefore, Eq. (2.1) is equivalent to the application of the operator $e^{-n_0\partial/\partial n}$ to amplitudes f_n in Eq. (2.2). Here we study accelerator modes, where the change of momentum by one iteration is $\pm 2\pi$. Therefore, in order to return an accelerator mode structure to its original position, we have to choose $n_0=\pm 2\pi/\tilde{h}$. We choose the mode with the initial angle $\theta_0=\pi/2$ (see Fig. 1) and the corresponding shift operator with $n_0=2\pi/\tilde{h}$ in Eq. (2.1). Therefore, in the numerical calculations the evolution operator is replaced by

$$\hat{J}\hat{U}=\exp\{-i\tilde{h}(\hat{n}+2\pi/\tilde{h})^2/2\}\times\exp\{-i(K/\tilde{h})\cos\theta-i2\pi\theta/\tilde{h}\}. \tag{2.3}$$

Parts of the wave function that are not trapped in the accelerator mode island do not propagate by $+2\pi$ in p per step. Therefore the action of the shift on those parts is not simply related to the dynamics of the part of the wave packet in the region Δp, where the survival probability is calculated. To avoid complications, these contributions are eliminated by the absorbing boundary conditions at the edges of the interval Δp. If in this region there are N states with $-(N+1)/2 \leqslant n \leqslant (N-1)/2$ the action of the projection operator $\hat{\mathcal{P}}$ on an angular momentum state with the amplitude f_n is

$$\hat{\mathcal{P}}f_n=\begin{cases} f_n: & -\dfrac{N+1}{2}\leqslant n\leqslant\dfrac{N-1}{2} \\ 0: & \text{otherwise.}\end{cases} \tag{2.4}$$

The quantum map used to follow the evolution of the wave packet due to the accelerator mode island is therefore

$$\Psi_{t+1}=\hat{\mathcal{P}}\hat{J}\hat{U}\Psi_t, \tag{2.5}$$

and the survival probability in the region Δp is

$$P(t)=|\Psi_t|^2. \tag{2.6}$$

Because of the form of the dynamics Eq. (2.5), this survival probability can be interpreted as the survival probability in a specific accelerator mode island, the one near $\theta_0=\pi/2$ in our case. The operator $\hat{\mathcal{P}}\hat{J}\hat{U}$ of Eq. (2.5) enables us to explore the survival probability of a wave packet in an accelerator mode island by following one island. Since the decay of the island survival probability is very slow, straightforward calculations of the evolution by $\hat{U}$ requires an extremely large basis [35]. The shift $\hat{J}$ keeps the island in its original position as explained following Eq. (2.2). In order to identify the occupation probability of the island with the one of the interval Δp, particles should be eliminated when they reach the boundary of the interval. This is precisely the effect of the absorbing boundary conditions. The evolution of the wave function starting from the initial state $n=0$, which is uniformly spread in θ, is presented in Figs. 2 and 3. We see that after many iterations only the part confined to the island survives. Therefore, $P(t)$ is indeed the survival probability inside the island.

The survival probability $P(t)$ was calculated from the initial state $n=0$ that overlaps the accelerator mode islands structure. The survival probability $P(t)$ as a function of time is presented in Figs. 4 and 5. The effective Planck's constant was taken as $\tilde{h}=2\pi/(N+g)$, where $g=(\sqrt{5}-1)/2$ is the inverse golden mean and N is in the interval $(5\times10^3,\ 2.5\times10^4)$. This choice of $\tilde{h}$ determines the number of states N in a classical unit cell of phase space, such that the classical limit $\tilde{h}\rightarrow 0$ has a transparent meaning of an infinite number of states in a cell. Another important property of this representation is irrationality of the reduced Planck's constant

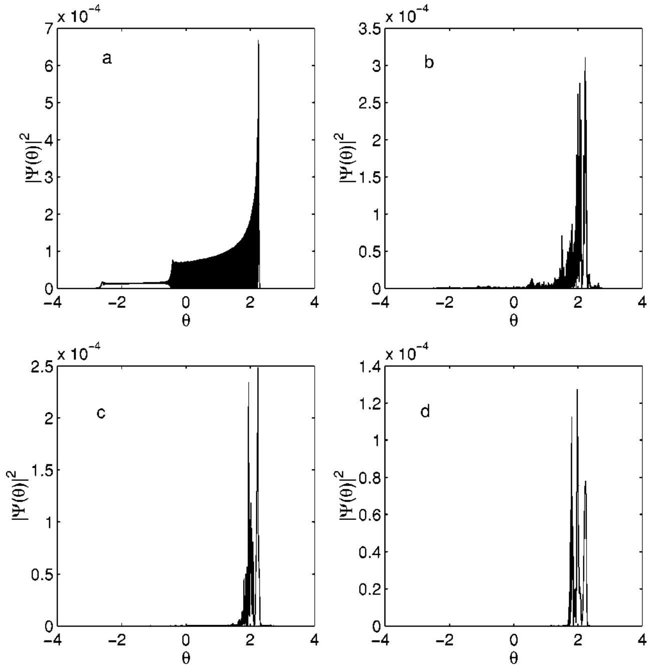

FIG. 2. The probability amplitude for the four times: (a) $t=1$, (b) $t=10$, (c) $t=100$, and (d) $t=1000$ iterations. The solution is obtained by iterations of the quantum map (2.5) for $K=K^{(1)}$ and $N=5557$ with the initial condition for $n=0$.

$\tilde{h}/2\pi$, so that the obvious quantum resonances [2,36] resulting in ballistic motion $\langle p^2\rangle \sim t^2$ are avoided. The value of the effective Planck's constant is controlled by the variation of N. The simulation time is sufficiently long so that it is longer than the Heisenberg time N or comparable to it. Different

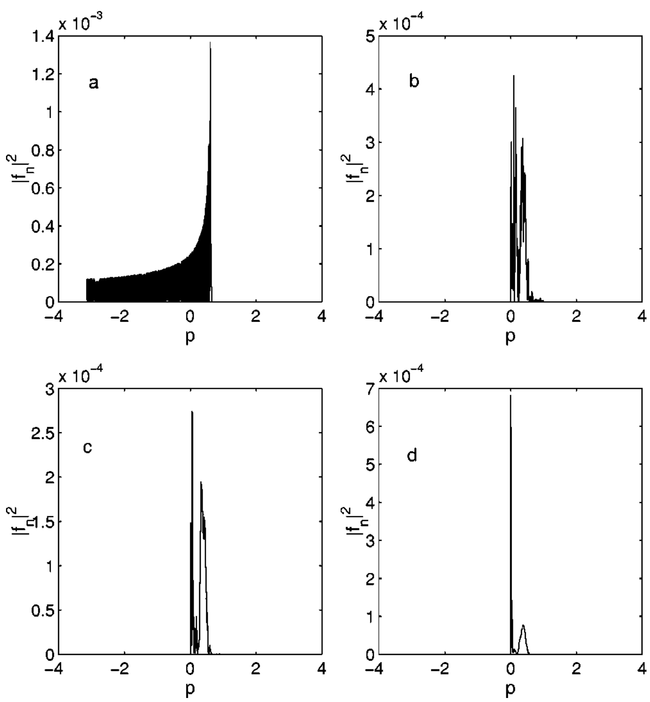

FIG. 3. The level occupation probability amplitude vs angular momentum corresponding to the wave functions presented in Fig. 2.

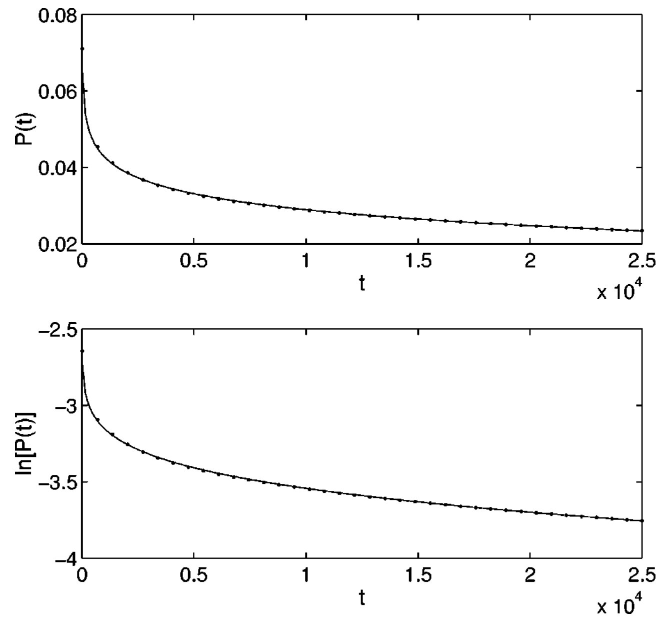

FIG. 4. Typical evolution of the quantum survival probability for $N=10\,557$ and $K=K^{(1)}$. The doted lines correspond to the numerical calculations and solid lines to the analytical formula (3.16).

behavior was found when the initial wave function does not overlap the accelerator mode island [31]. It will be discussed at the end of the next Sec. III.

III. QUANTUM TRANSPORT FOR THE ACCELERATOR MODE STRUCTURES: ANALYTICAL ESTIMATE

We now calculate the tunneling from an accelerator mode island within some approximations. The map Eq. (1.2) can be expanded in the small vicinity of the accelerator modes $(p_0,\theta_0)=(2\pi m,\pm\pi/2)$ as [14]

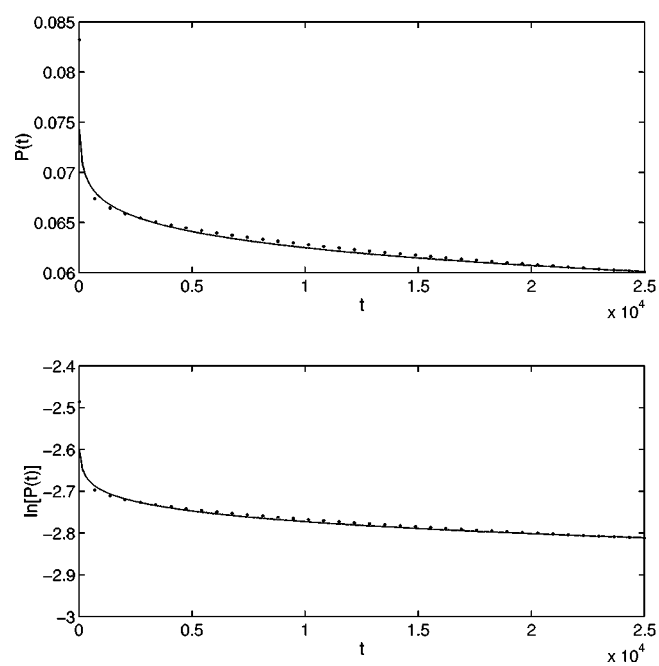

FIG. 5. The same as Fig. 4, for $K=K^{(2)}$.

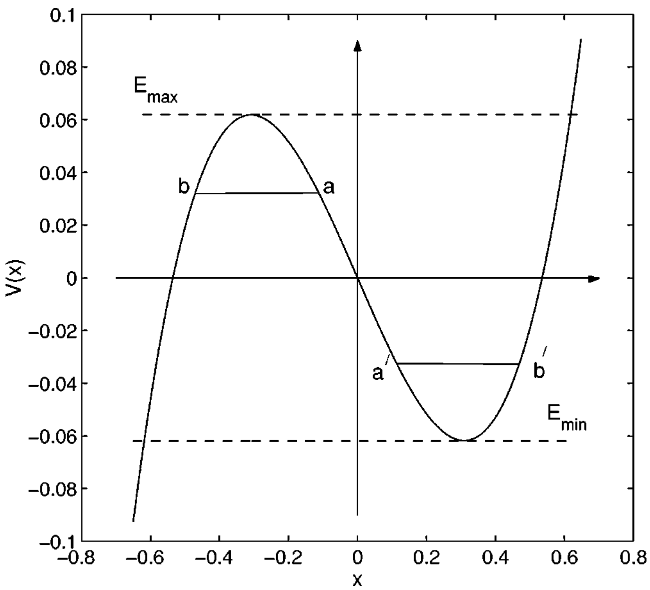

FIG. 6. The potential $V(x)$

$$\delta p' = \delta p \pm \Delta_K - (\pm)\pi \delta\theta^2,$$

$$\delta\theta' = \delta\theta + \delta p', \tag{3.1}$$

where $\Delta_K = K - 2\pi k$, $\delta p = p - 2\pi m$, and $\delta\theta = \theta - (\pm \pi/2)$. For the sake of definiteness we confine ourselves to $\theta = \pi/2$ (the result for $\theta = -\pi/2$ is identical). The differences between $\delta p'$ and δp and between $\delta\theta'$ and $\delta\theta$ are very small, therefore the map Eq. (3.1) can be approximated by the Hamilton equations generated by the Hamiltonian

$$H_{\mathrm{acc}} = p^2/2 + V(x), \tag{3.2}$$

where δp is replaced by $p, \delta\theta$ by x, and the potential is

$$V(x) = -\Delta_K x + \frac{\pi}{3} x^3 \tag{3.3}$$

and is depicted in Fig. 6. The Hamiltonian, (3.2) with the potential $V(x)$ of Eq. (3.3) was obtained in [8] and in a more general way in [9], while the topology of islands produced by the Hamiltonian was studied in [14]. In the semiclassical Wentzel-Kramers-Brillouin approximation the tunneling probability per unit time of a particle with energy E out of the potential well is

$$\gamma(E) = \exp\{-S_{\mathrm{ins}}(E)/\tilde{\hbar}\}/T(E), \tag{3.4}$$

where $S_{\mathrm{ins}}(E) = \int_a^b \sqrt{2(V(x)-E)}dx$ is the instanton action, the points a and b are the limits (turning points) of the motion under the barrier (see Fig. 6), and $T(E)$ is the period of the classical motion with energy E in the well. The survival probability at the energy E is

$$P_E(t) = e^{-\gamma(E)t} P_E(t=0). \tag{3.5}$$

The resulting total survival probability is

$$P(t) = \int_{E_{\min}}^{E_{\max}} dE\, \rho(E) \cdot e^{-\gamma(E)t} P_E(t=0), \tag{3.6}$$

where $\rho(E)$ is the density of states at energy E, while $E_{\max}$ and $E_{\min}$ are the maximal and minimal energies of states that are trapped in the well. It was assumed that this semiclassical formula holds for states as high as $E_{\max}$. This will not introduce a large error because the states in the vicinity of $E_{\max}$ decay vary fast. The potential $V(x)$ of Eq. (3.3) satisfies $V(-x) = -V(x)$, therefore,

$$\begin{aligned} S_{\mathrm{ins}}(E) &= 2\int_a^b \sqrt{2(V(x)-E)}dx \\ &= 2\int_{a'}^{b'} \sqrt{2[(-E)-V(-x)]}dx \\ &= 2\int_{a'}^{b'} p(-E)dx = S(-E), \end{aligned} \tag{3.7}$$

where $S(-E)$ is the action of a periodic orbit of energy $-E$ in the well (see Fig. 6). The symmetry of the potential also implies $E_{\max} = -E_{\min} \equiv E_0$. The semiclassical density of states is

$$\rho(E) = \frac{1}{2\pi\tilde{\hbar}} \frac{dS(E)}{dE} = \frac{T(E)}{2\pi\tilde{\hbar}}. \tag{3.8}$$

Consequently, the survival probability Eq. (3.6) takes the form

$$\begin{aligned} P(t) = \frac{1}{2\pi\tilde{\hbar}} \int_{-E_0}^{E_0} dE \left(\frac{dS}{dE}\right) \\ \times \exp[-te^{-S(-E)/\tilde{\hbar}}/T(E)] P_E(t=0). \end{aligned} \tag{3.9}$$

Since there is no general relation between $S(E)$ and $S(-E)$, in order to make progress, an approximation for $S(E)$ will be introduced. It will be approximated by the linear function of energy

$$S(E) \approx S_0 + T_0 E, \tag{3.10}$$

where $S_0 \equiv S(E=0)$ and

$$T_0 = T(E=0) = \frac{dS}{dE}\Big|_{E=0}$$

are the action and the period for $E=0$. Within this approximation the period is approximated by its value at zero energy $T(E) \approx T_0$. This approximation should be reasonable for a wide range of energy since even at the separatrix the dependence of the period on energy is logarithmic, namely $T(E) \sim \ln(E_0 - E)$. Within the linear approximation of the action Eq. (3.10), the survival probability (3.9) takes the form

$$P(t)=\frac{T_0}{2\pi\tilde{h}}\int_{-E_0}^{E_0}dE\,\exp\left[-\frac{t}{T_0}e^{-S_0/\tilde{h}}e^{T_0E/\tilde{h}}\right]P_E(t=0). \tag{3.11}$$

In order to complete the calculation we have to make some assumptions on the initial distribution. We will assume first that $P_E(t=0)=P^0$, which is independent of energy. Then the integral (3.11) can be evaluated by the change of variable to $y=e^{T_0E/\tilde{h}}$, namely,

$$P(t)=\frac{P^0}{2\pi}\int_{y_-}^{y_+}\frac{dy}{y}\exp\left[-\frac{ty}{T_0}e^{-S_0/\tilde{h}}\right], \tag{3.12}$$

where $y_\pm=e^{\pm T_0E_0/\tilde{h}}$. Therefore,

$$P(t)=\frac{P^0}{2\pi}\left[E_1\left(\frac{t}{T_0}e^{-(S_0+T_0E_0)/\tilde{h}}\right)-E_1\left(\frac{t}{T_0}e^{-(S_0-T_0E_0)/\tilde{h}}\right)\right] \tag{3.13}$$

and within the linear approximation

$$P(t)=\frac{P^0}{2\pi}\left[E_1\left(\frac{t}{T_0}e^{-S(E_0)/\tilde{h}}\right)-E_1\left(\frac{t}{T_0}e^{-S(-E_0)/\tilde{h}}\right)\right]. \tag{3.14}$$

Here $E_1(z)=\int_z^\infty dy\,e^{-y}/y$ is the exponential integral [37]. Note that the arguments of the two terms in Eq. (3.14) can be very different, their ratio is $\exp\{[S(E_0)-S(-E_0)]/\tilde{h}\}=e^{2T_0E_0/\tilde{h}}$. For small values of its argument the exponential integral behaves as $E_1(z)\approx -C+\ln(1/z)$, where C is Euler's constant while the large argument asymptotic behavior is $E_1(z)\sim e^{-z}/z$. Therefore, for

$$t/T_0 \gg e^{S(-E_0)/\tilde{h}} \tag{3.15}$$

the result Eq. (3.14) for the survival probability simplifies and takes the form

$$P(t)=aE_1(c(\tilde{h})t), \tag{3.16}$$

with

$$c(\tilde{h})=c_1e^{-c_2/\tilde{h}}. \tag{3.17}$$

For the specific potential (3.3) used in the calculation: $a=P^0/2\pi$, while $c_1=1/T_0$ and $c_2=S(-E_0)$. The calculation presented here was done for the specific potential (3.3) that is the relevant one for the kicked rotor. Also for other systems the accelerator mode islands are described by cubic potentials [9,10] and in this sense the classical dynamics is universal, therefore the form Eq. (3.16) with Eq. (3.17) of the survival probability is expected to be universal as well. If the approximation (3.15) cannot be made, and the second term in Eq. (3.14) cannot be neglected, then Eq. (3.16) should be replaced by

$$P(t)=aE_1(c(\tilde{h})t)-a'E_1(c'(\tilde{h})t), \tag{3.18}$$

where the dependence of $c'(\tilde{h})$ on $\tilde{h}$ is similar to Eq. (3.17).

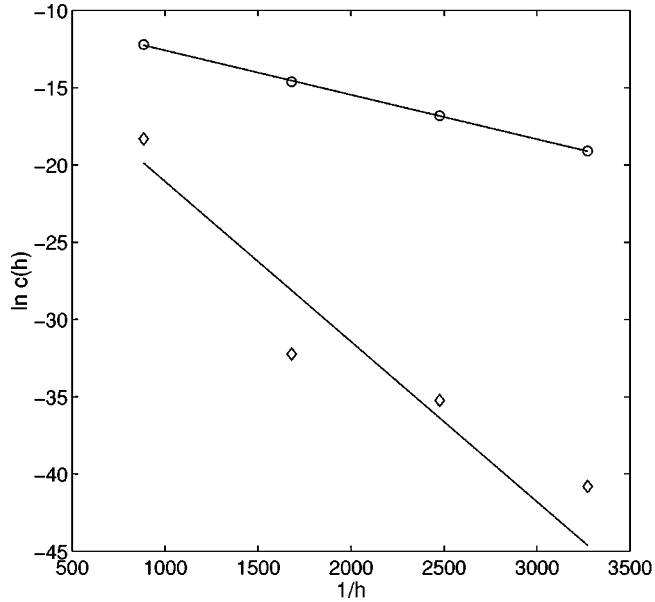

FIG. 7. The tunneling coefficients $c(\tilde{h})$ vs $1/\tilde{h}$, where (○) mark the result of numerical calculations for $K=K^{(1)}$ and (◇) for $K=K^{(2)}$. The solid lines are the slopes obtained by the least squares fit.

These predictions are compared with the survival probability inside the accelerator mode island calculated numerically for the initial state $n=0$, that has large overlap with the accelerator mode islands. Representative results are shown in Figs. 4 and 5, where the numerical simulations are compared with the analytical prediction Eq. (3.16), where the parameters a and $c(\tilde{h})$ were fitted. For $K^{(1)}$ these were found to take the values $a=0.006$ and $c(\tilde{h})=5\times10^{-7}$ for $N=10\,557$, and the contribution of the second term in Eq. (3.18) is negligible. For $K^{(2)}$ the values $a=0.0028$, $c(\tilde{h})=10^{-14}$ and $a'=0.000\,64$, $c'(\tilde{h})=10^{-3}$ for $N=10\,557$ were obtained. In order to check Eq. (3.17), $\tilde{h}(N)$ was varied and the results are presented in Fig. 7. The least squares fit results in $c_1=60\times10^{-5}$ and $c_2=0.0029\pm0.000\,03$ for $K^{(1)}$, while $c_1=2.25\times10^{-5}$ and $c_2=0.0104\pm0.001\,9$ for $K^{(2)}$. From these results one concludes that for $K^{(1)}$ the potential barrier is more penetrable to quantum tunneling than for $K^{(2)}$. We conclude that for quantum systems, initially populated in the accelerator mode islands, the survival probability satisfies Eq. (3.16), therefore,

$$P(t)=a\begin{cases}-C-\ln c(\tilde{h})t: & t\ll 1/c(\tilde{h})\\ e^{-c(\tilde{h})t}/c(\tilde{h})t: & t\gg 1/c(\tilde{h})\end{cases} \tag{3.19}$$

and $c(\tilde{h})$ is given by Eq. (3.17).

It is reasonable to anticipate the same asymptotic behavior of $P(t)$ for the initial population in the chaotic region, because the exponential localization of the wave packet as well as the localization length should be independent of choice of the initial conditions. This result is because wave functions on the accelerator mode islands structures are coupled to the ones in the chaotic sea via tunneling, and therefore, they are expanded in terms of the same set of eigenfunctions. The asymptotic decay is dominated by the eigenfunctions with the slowest decay. Consequently, the exponential decay of the survival probability for the initial population in the chaotic region should be the same as in Eq. (3.19) for asymptotically long times $(t \gg 1/c(\tilde{h}))$. Indeed, for a time shorter than the quantum–classical crossover, explored in [21,31], classical sticking takes place. For longer times the trapping is quantum mechanical and the survival probability is given by Eq. (3.11) but the initial distribution is proportional to the probability of tunneling through the barrier, namely,

$$P_E(t=0) = Ae^{-S_{\text{ins}}(E)/\tilde{h}}, \tag{3.20}$$

where A is a constant. The reason is that for this initial preparation the occupation probability of a state is determined by its coupling to the chaotic component [38]. Using Eq. (3.10) one obtains

$$P(t) = \frac{AT_0}{2\pi\tilde{h}} \int_{-E_0}^{E_0} dE \exp\left[-\frac{t}{T_0} e^{-S_0/\tilde{h}} e^{T_0 E/\tilde{h}} \right] e^{-S_0/\tilde{h}} e^{T_0 E/\tilde{h}}. \tag{3.21}$$

Changing the integration variable to $y = e^{-S_0/\tilde{h}} e^{T_0 E/\tilde{h}}$ one finds

$$P(t) = \frac{A}{2\pi} \int_{y'_-}^{y'_+} dy \, e^{-ty/T_0}, \tag{3.22}$$

where $y'_{\pm} = e^{-S_0/\tilde{h}} e^{\pm T_0 E/\tilde{h}}$, resulting in

$$P(t) = \frac{A}{2\pi} \cdot \frac{T_0}{t} \left[\exp\left\{ -\frac{t}{T_0} e^{-S(E_0)/\tilde{h}} \right\} - \exp\left\{ -\frac{t}{T_0} e^{-S(-E_0)/\tilde{h}} \right\} \right]. \tag{3.23}$$

For $t/T_0 \gg e^{S(-E_0)/\tilde{h}}$ the second term is negligible, resulting in

$$P(t) = be^{-c(\tilde{h})t}/t, \tag{3.24}$$

where $c(\tilde{h})$ is given by Eq. (3.17). Comparing this result to Eq. (3.19) we note that the long time $(t \gg 1/c(\tilde{h}))$ asymptotic behavior for these two types of initial conditions, one in the accelerator mode island and the other in the chaotic region, is identical. For short time $t \ll 1/c(\tilde{h})$ the decay of wave packets started in the chaotic component is much faster, that is,

$$P(t) \sim b/t, \tag{3.25}$$

in agreement with the result of [31]. In addition, particles that were initially in the accelerator mode island may tunnel out to the chaotic component and then back into accelerator mode island. Consequently their survival probability behaves similarly to the survival probability of particles that were started in the chaotic component.

IV. THE QUASIENERGY STATES

In this section a form of the quasienergy states implied by the transport found in Secs. II and III is proposed. Any wave function can be expanded as

$$\psi(\theta,t) = \sum_{\omega} C_\omega \psi_\omega(\theta,t), \tag{4.1}$$

where $\psi_\omega(\theta,t)$ are the quasienergy states and the expansion coefficients are $C_\omega = \langle \psi_\omega(\theta,t=0) | \psi(\theta,t=0) \rangle$. Therefore, localization in transport and localization of the quasienergy states are closely related. In Sec. III the survival probability on accelerator mode islands was calculated as a function of time. Since in one time step a trajectory in the accelerator mode island structure travels a distance 2π in p or $2\pi/\tilde{h}$ in n, in a time t the distance it travels in n is $n = (2\pi/\tilde{h})t$. The resulting spread in n can be inferred from the results of the previous section simply by the substitution of $\tilde{h}n/2\pi$ for t. Consequently, for long distances one concludes from Eqs. (3.19) and (3.24) that the occupation probability decays as

$$\tilde{P}(n) \sim e^{-2n/\xi}/n \tag{4.2}$$

with

$$\xi = 4\pi/\tilde{h}c(\tilde{h}) = (4\pi/c_1)e^{c_2/\tilde{h}}/\tilde{h}. \tag{4.3}$$

The prefactor cannot be determined reliably from such heuristic considerations. The quasienergy states are expected to decay exponentially with the localization length ξ of Eq. (4.3). For small $\tilde{h}$, where our results are derived, it is much larger than the localization length Eq. (1.8) resulting from suppression of normal diffusion by interference. For the behavior on distances shorter than ξ, where the decay of the survival probability is weaker than exponential, the decay of wave functions as a function of n is weaker than the exponential as well. The predictions in this regime are different for the two types of initial conditions considered in the previous Sec. III. In general one therefore expects, for distances smaller than the localization length, that the distribution behaves as

$$\tilde{P}(n) = A_1[-C - \ln(n/\xi)] + A_2/n, \tag{4.4}$$

where the two contributions result from Eqs. (3.19) and (3.24). The constants A_1 and A_2 are determined by the initial conditions.

It should be noted that formulas (1.8) and (4.3) do not relate to each other, and there is no contradiction between them. The $\tilde{h}$ scaling of the localization length according Eq. (1.8), namely $\xi \sim 1/\tilde{h}^2$, was also observed for anomalous diffusion with large (in the quantum limit) values of $\tilde{h}$ as well [21,22,26]. In this case formula (4.3) cannot be applied, because the values of $\tilde{h}$ are larger than the accelerator mode island area, and it is impossible to "put" even one level inside the island structure. In the opposite regime, when $\tilde{h}$ is small enough to make it possible to contain a few levels, at least inside the structure, a part of the wave packet spreads as the accelerator mode does. In this case the present analysis is valid and formula (4.3) for the $\tilde{h}$ scaling of the localization length holds.

Because different quasienergies may exhibit different behavior related to the the values of A_1 and A_2, it is reasonable to speculate that their distribution depends only on n/ξ, since their behavior is dominated by tunneling and ξ is the scale determined by this process. This should hold if n is sufficiently large so that classical trapping by the boundary islands chain is not important. For short distances, shorter than those corresponding to time scales shorter than $\tau_{\hbar} \sim (1/\tilde{h})^{\mu}$ [the time scale corresponding to quantum classical crossover [31] specified by the transport exponent μ in Eq. (1.6)], the trapping by the boundary islands chain is of classical origin. This results in various power law decays on the very short distances, shorter than $2\pi\tau_{\hbar}/\tilde{h}$, related to the decay of the classical survival probability of particles trapped in the boundary islands chain, relevant for the specific value of K that is considered. A crucial assumption made in this work is that the irrationality of $\tilde{h}$ prevents resonant tunneling between the accelerator mode islands. This assumption still requires justification. It is expected to hold for the same reasons it holds for the other values of K [7,19].

V. SUMMARY AND DISCUSSION

In this paper the decay of wave packets by tunneling out of accelerator mode islands was calculated analytically for a simple model potential (3.3) in the semiclassical limit where the effective Planck's constant $\tilde{h}$ is much smaller than the islands. For this model, analytical expressions for survival probability were obtained for situations where the initial wave packet is inside the island Eq. (3.14) and in the chaotic region Eq. (3.23). With the help of the modified evolution operator (2.5) the accelerator mode islands could be followed for a very long time. In this way the decay of the survival probability of the wave packet in an island could be computed and the form of the analytical expression Eq. (3.14) could be verified. The result Eq. (3.25) for the wave packet starting in the chaotic region was verified in [31] using the same numerical method for a relatively short time. It is important to notice that the accelerator mode islands with the chain structure around them form a much more complicated phase space structure than the model potential (3.3), but the qualitative behavior of the survival probability is similar. This demonstrates its robustness. The tunneling results in the decay of the classical accelerator modes. It sets a time scale $1/c(\tilde{h})$ of Eq. (3.17), leading to a localization length ξ of Eq. (4.3) for this process. Transitions between the chaotic region and the accelerator modes take place by tunneling. Localization that ultimately takes place [39] results in exponential localization on a length scale on the order of ξ and should be the subject of further studies.

This work demonstrates how a classical structure in a mixed system (a system where part of the phase space is chaotic and part is regular) decays as a result of tunneling. If the effective Planck's constant ($\hbar$ divided by the natural action of the system) is small, this time can be extremely long, as is the case in the present work.

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[1] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
[2] B.V. Chirikov, Phys. Rep. **52**, 263 (1979).
[3] A.B. Rechester and R.B. White, Phys. Rev. Lett. **44**, 1586 (1980).
[4] R. Balescu, *Statistical Dynamics, Matter out of Equilibrium* (Imperial College Press, Singapore, 1997).
[5] M. Khodas and S. Fishman, Phys. Rev. Lett. **84**, 2837 (2000); M. Khodas, S. Fishman and O. Agam, Phys. Rev. E **62**, 4769 (2000).
[6] G. Casati, B. V. Chirikov, F. M. Izrailev and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, edited by G. Casati and J. Ford (Springer, New York, 1979).
[7] S. Fishman, D.R. Grempel, and R.E. Prange, Phys. Rev. Lett. **49**, 509 (1982); D.R. Grempel, R.E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984).
[8] C.F.F. Karney, Physica D **8**, 360 (1983).
[9] V. K. Melnikov, in *Transport, Chaos and Plasma Physics, II, Proceedings, Marseille*, edited by F. Doveil, S. Benkadda, and Y. Elskens (World Scientific, Singapore, 1996), p. 142.
[10] V. Rom-Kedar and G.M. Zaslavsky, Chaos **9**, 697 (1999).
[11] S. Newhous, Am. J. Math. **99**, 1061 (1977).
[12] J.D. Meiss *et al.*, Physica D **6**, 375 (1983).
[13] J.D. Meiss, Phys. Rev. A **34**, 2375 (1986); Rev. Mod. Phys. **64**, 795 (1992).
[14] G.M. Zaslavsky, M. Edelman, and B.A. Niyazov, Chaos **7**, 159 (1997).
[15] G.M. Zaslavsky and M. Edelman, Chaos **10**, 135 (2000).
[16] S. Benkadda, S. Kassibrakis, R. White, and G. Zaslavsky,

Phys. Rev. E **55**, 4909 (1997); **59**, 3761 (1999).
[17] J.D. Hanson, E. Ott, and T.M. Antonsen, Phys. Rev. A **29**, 819 (1984).
[18] B.V. Chirikov, F.M. Izrailev, and D.L. Shepelyansky, Sov. Sci. Rev. **C2**, 209 (1981).
[19] D. Shepelyansky, Physica D **28**, 103 (1987).
[20] S. Fishman, R.E. Prange, and M. Griniasty, Phys. Rev. A **39**, 1628 (1989).
[21] B. Sundaram and G.M. Zaslavsky, Phys. Rev. E **59**, 7231 (1999).
[22] A. Iomin and G.M. Zaslavsky, Chaos **10**, 147 (2000).
[23] A. Iomin and G.M. Zaslavsky, Phys. Rev. E **60**, 7580 (1999).
[24] R. Roncaglia, L. Bonci, B.J. West, and P. Grigolini, Phys. Rev. E **51**, 5524 (1995); L. Bonci, P. Grigolini, A. Laux, and R. Roncaglia, Phys. Rev. A **54**, 112 (1996).
[25] A. Iomin and G. M. Zaslavsky (unpublished).
[26] R. Artuso and M. Rusconi, Phys. Rev. E **64**, 015204(R) (2001).
[27] Numerical studies of quantum localization of anomalous diffusion in a discontinuous map were recently reported in T. Prosen and M. Z̃nidaric̃ (unpublished).
[28] B.G. Klappauf, W.H. Oskay, D.A. Steck, and M.G. Raizen, Phys. Rev. Lett. **81**, 4044 (1999).
[29] S. Fishman, D.R. Grempel, and R.E. Prange, Phys. Rev. A **36**, 289 (1987); D.R. Grempel, S. Fishman, and R.E. Prange, Phys. Rev. Lett. **53**, 1212 (1984).
[30] Y.-C. Lai, E. Ott, and C. Grebogi, Phys. Lett. A **173**, 148 (1993).
[31] A. Iomin and G.M. Zaslavsky, Phys. Rev. E **63**, 047203 (2001).
[32] R.B. Shirts and W.P. Reinhardt, J. Chem. Phys. **77**, 5204 (1982).
[33] T. Geisel, G. Radons, and J. Rubner, Phys. Rev. Lett. **57**, 2883 (1986).
[34] R.S. MacKay and J.D. Meiss, Phys. Rev. A **37**, 4702 (1988).
[35] The size of the basis is so large, that it cannot be handled even by present supercomputers.
[36] F.M. Izrailev and D.L. Shepelyansky, Dokl. Akad. Nauk UzSSR **249**, 1103 (1979); Ther. Mat. Fiz. **43**, 417 (1980).
[37] M. Abramovitz and I. Stegun, *Handbook of Mathematical Functions* (Washington, 1968).
[38] R. Ketzmerick, L. Hufnagel, F. Steinbach, and M. Weiss (unpublished).
[39] E. Ott and R. E. Prange (private communication).